Inverse of a Matrix

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Inverse of a Matrix

- What is the inverse of a matrix?
- Elementary matrix
- What kinds of matrices are invertible
- Find the inverse of a general invertible matrix

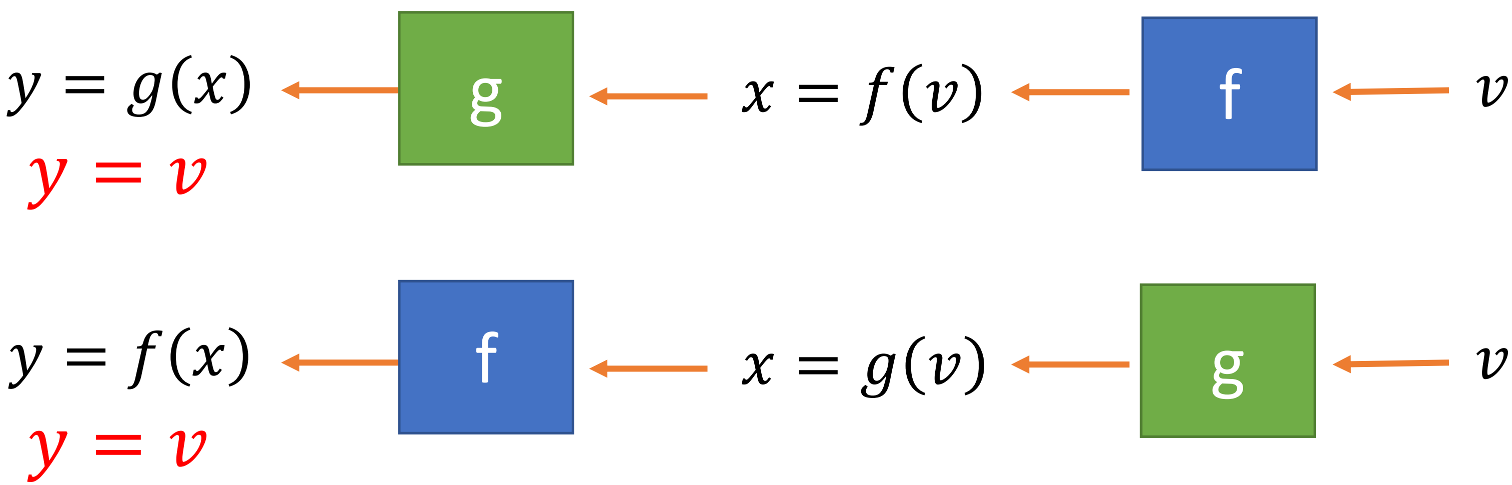
Inverse of a Matrix

What is the inverse
of a matrix?

Inverse of Function

- Two function f and g are inverse of each other ($f=g^{-1}$, $g=f^{-1}$) if ……

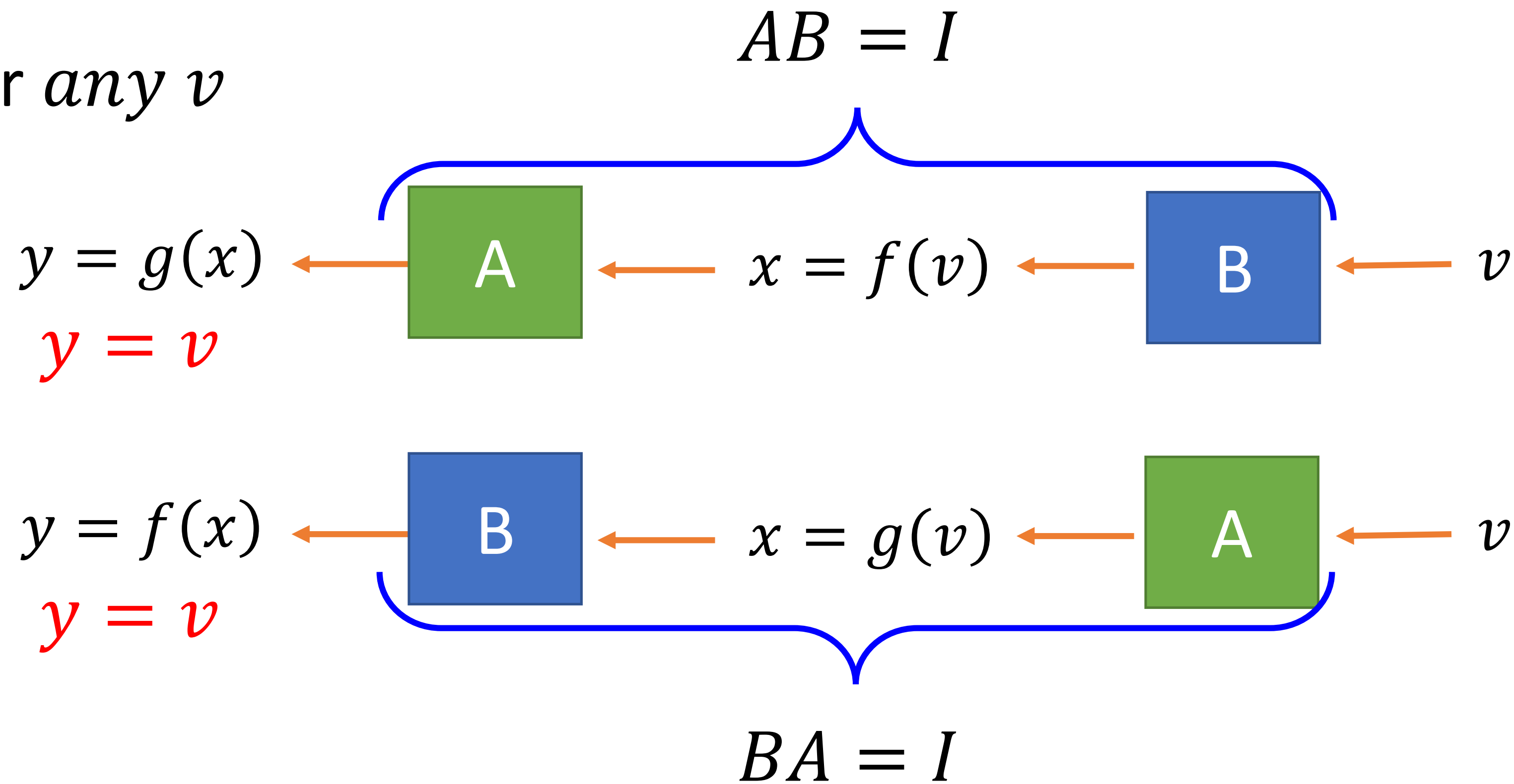
For $any\ v$



Inverse of Matrix

- If B is an inverse of A, then A is an inverse of B, i.e., A and B are inverses to each other.

For $any\ v$



Inverse of Matrix

- If B is an inverse of A, then A is an inverse of B, i.e., A and B are inverses to each other.

A is called invertible if there is a matrix B such that $AB = I$ and $BA = I$

B is an inverse of A

$$B = A^{-1} \quad A^{-1} = \mathrm{B}$$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \quad B = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Inverse of Matrix

- If B is an inverse of A, then A is an inverse of B, i.e., A and B are inverses to each other.

n x n?

A is called invertible if there is an matrix B such that $AB = I$ and $BA = I$

B is an inverse of A

$$B = A^{-1} \quad A^{-1} = \mathrm{B}$$

Non-square matrix cannot be invertible

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & 1 \\ -1 & -1 \\ 0 & 2 \end{bmatrix}.$$

Inverse of Matrix

- Not all the square matrix is invertible

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- Unique

$$AB = I \qquad BA = I \qquad AC = I \qquad CA = I$$

$$B = BI = B(AC) = (BA)C = IC = C$$

Solving Linear Equations

- The inverse can be used to solve system of linear equations.

$$A\mathbf{x} = \mathbf{b}$$

$$\underbrace{\begin{aligned} x_1 + 2x_2 &= 4 \\ 3x_1 + 5x_2 &= 7 \end{aligned}}_{Ax = b}$$

If A is invertible.

$$A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b}$$

$$\mathbf{x} = A^{-1}\mathbf{b}$$

$$= \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} -6 \\ 5 \end{bmatrix}$$

However, this method is computationally inefficient.

Input-output Model

- 假設世界上只有食物、黃金、木材三種資源

	需要食物	需要黃金	需要木材
生產一單位食物	0.1	0.2	0.3
生產一單位黃金	0.2	0.4	0.1
生產一單位木材	0.1	0.2	0.1

$$
\begin{array}{ccccc}
\text{Cx} & & \text{C} & & \text{x} \\
\begin{bmatrix} 0.1x_1 + 0.2x_2 + 0.1x_3 \\ 0.2x_1 + 0.4x_2 + 0.2x_3 \\ 0.3x_1 + 0.1x_2 + 0.1x_3 \end{bmatrix} & = & \begin{bmatrix} 0.1 & 0.2 & 0.1 \\ 0.2 & 0.4 & 0.2 \\ 0.3 & 0.1 & 0.1 \end{bmatrix} & & \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
\text{須投入} & & \text{Consumption matrix} & & \text{想生產}
\end{array}
$$

Input-output Model

$$\underset{\text{須投入}}{\overset{Cx}{\begin{bmatrix} 48 \\ 96 \\ 53 \end{bmatrix}}} = \underset{\text{Consumption matrix}}{\overset{C}{\begin{bmatrix} 0.1 & 0.2 & 0.1 \\ 0.2 & 0.4 & 0.2 \\ 0.3 & 0.1 & 0.1 \end{bmatrix}}} \underset{\text{想生產}}{\overset{x}{\begin{bmatrix} 100 \\ 150 \\ 80 \end{bmatrix}}}$$

須考慮成本：

淨收益

$$x - Cx = \begin{bmatrix} 100 \\ 150 \\ 80 \end{bmatrix} - \begin{bmatrix} 48 \\ 96 \\ 53 \end{bmatrix} = \begin{bmatrix} 52 \\ 54 \\ 27 \end{bmatrix}$$

Demand Vector d

Input-output Model

$$C = \begin{bmatrix} 0.1 & 0.2 & 0.1 \\ 0.2 & 0.4 & 0.2 \\ 0.3 & 0.1 & 0.1 \end{bmatrix} \qquad d = \begin{bmatrix} 90 \\ 80 \\ 60 \end{bmatrix}$$

Demand Vector d

生產目標 x 應該訂為多少?

$$x - Cx = d$$

$$Ix - Cx = d$$

$$(I - C)x = d$$

Ax=b

$$\mathrm{A} = I - \mathrm{C} = \begin{bmatrix} 0.9 & -0.2 & -0.1 \\ -0.2 & 0.6 & -0.2 \\ -0.3 & -0.1 & 0.9 \end{bmatrix}$$

$$b = \begin{bmatrix} 90 \\ 80 \\ 60 \end{bmatrix} \qquad x = \begin{bmatrix} 170 \\ 240 \\ 150 \end{bmatrix}$$

Input-output Model

- 提升一單位食物的淨產值，需要多生產多少資源？

Ans: The first column of $(I-C)^{-1}$

$$(I-C)x = d \Rightarrow x = (I-C)^{-1}d$$

$$d \Rightarrow d + \begin{bmatrix}1\\0\\0\end{bmatrix} = d + e_1$$

$$x' = (I-C)^{-1}(d+e_1)$$

$$= (I-C)^{-1}d + \underline{(I-C)^{-1}e_1}$$

$$(I-C)^{-1} = \begin{array}{c} \begin{bmatrix}1.3 & 0.475 & 0.25\\0.6 & 1.950 & 0.50\\0.5 & 0.375 & 1.25\end{bmatrix} \\ \begin{array}{ccc}\text{食物} & \text{黃金} & \text{木材}\end{array}\end{array}$$

Inverse for matrix product

- A and B are invertible nxn matrices, is AB invertible? yes

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$B^{-1}A^{-1}(AB) \;= B^{-1}\,(A^{-1}A)B \;= B^{-1}\,B \;= I$$

$$(AB)B^{-1}A^{-1} \;= A(BB^{-1})A^{-1} \;= A\,A^{-1} \;= I$$

- Let $A_1, A_2, \cdots, A_k$ be nxn invertible matrices. The product $A_1 A_2 \cdots A_k$ is invertible, and

$$(A_1 A_2 \cdots A_k)^{-1} = (A_k)^{-1}(A_{k-1})^{-1} \cdots (A_1)^{-1}$$

Inverse for matrix transpose

- If A is invertible, is A^T invertible?

$$(A^T)^{-1} =? \quad (A^{-1})^T$$

$$\boxed{(AB)^T = B^T A^T}$$

$$A^{-1}A = I \Rightarrow (A^{-1}A)^T = I \Rightarrow A^T(A^{-1})^T = I$$

$$AA^{-1} = I \Rightarrow (AA^{-1})^T = I \Rightarrow (A^{-1})^T A^T = I$$

Inverse of a Matrix

Inverse of elementary matrices

Elementary Row Operation

- Every elementary row operation can be performed by matrix multiplication.

elementary matrix

- 1. Interchange

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

- 2. Scaling

$$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ kc & kd \end{bmatrix}$$

- 3. Adding *k* times row i to row j:

$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ ka+c & kb+d \end{bmatrix}$$

Elementary Matrix

- Every elementary row operation can be performed by matrix multiplication.
- How to find elementary matrix?

elementary matrix

E.g. the elementary matrix that exchange the 1st and 2nd rows

$$E\begin{bmatrix}1 & 4\\2 & 5\\3 & 6\end{bmatrix}=\begin{bmatrix}2 & 5\\1 & 4\\3 & 6\end{bmatrix}$$

$$E\cancel{\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}}=\begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}$$

$$\Rightarrow E=\begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}$$

Elementary Matrix

- How to find elementary matrix?
- Apply the desired elementary row operation on Identity matrix

Exchange the 2nd and 3rd rows $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Multiply the 2nd row by -4 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Adding 2 times row 1 to row 3 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

Elementary Matrix

- How to find elementary matrix?
- Apply the desired elementary row operation on Identity matrix

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$E_1 A =$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$E_2 A =$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_3 A =$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Inverse of Elementary Matrix

Reverse elementary row operation

Exchange the 2nd and 3rd rows

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Longleftrightarrow E_1^{-1} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Exchange the 2nd and 3rd rows

Multiply the 2nd row by -4

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Longleftrightarrow E_2^{-1} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Multiply the 2nd row by -1/4

Adding 2 times row 1 to row 3

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \Longleftrightarrow E_3^{-1} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Adding -2 times row 1 to row 3

RREF v.s. Elementary Matrix

- Let A be an mxn matrix with reduced row echelon form R.

$$E_k \cdots E_2 E_1 A = R$$

- There exists an invertible m x m matrix P such that PA=R

$$P = E_k \cdots E_2 E_1$$

$$P^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$

Inverse of a Matrix

Invertible

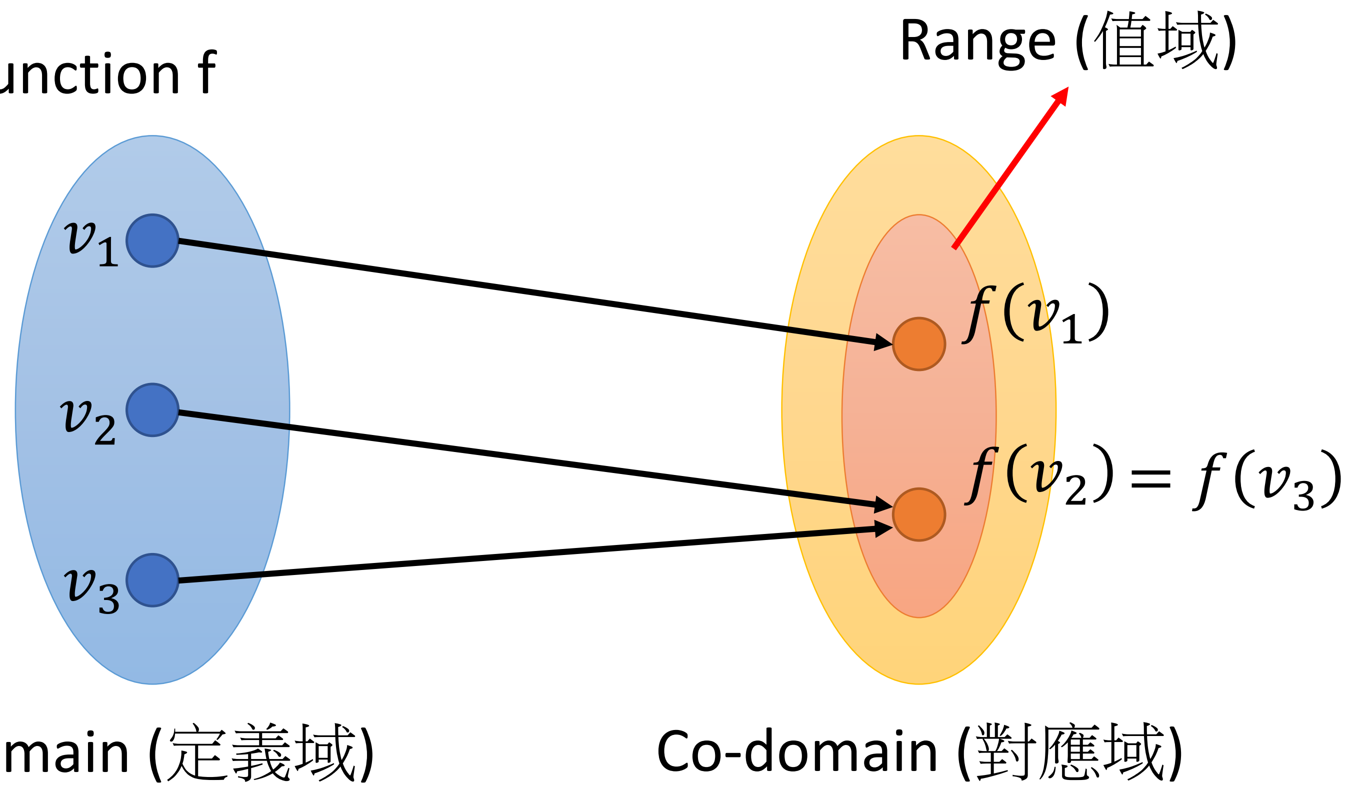
Summary

- Let A be an n x n matrix. A is invertible if and only if
  - The columns of A span R^n
  - For every b in R^n, the system Ax=b is consistent
  - The rank of A is n
  - The columns of A are linear independent
  - The only solution to Ax=0 is the zero vector
  - The nullity of A is zero
  - The reduced row echelon form of A is I_n
  - A is a product of elementary matrices
  - There exists an n x n matrix B such that $BA = I_n$
  - There exists an n x n matrix C such that $AC = I_n$

http://goo.gl/z3J5Rb

Review

- Given a function f

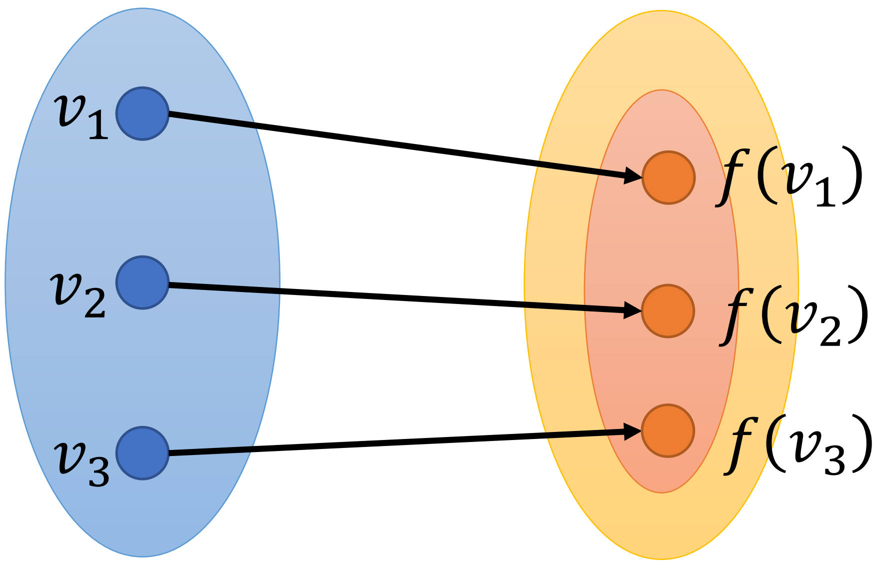


Given a linear function corresponding to a mxn matrix A

Domain=R^n Co-domain=R^m Range=?

One-to-one

- A function f is one-to-one



~~$f(x) = b$ has one solution~~

$f(x) = b$ has at most one solution

If co-domain is “smaller” than the domain, f cannot be one-to-one.

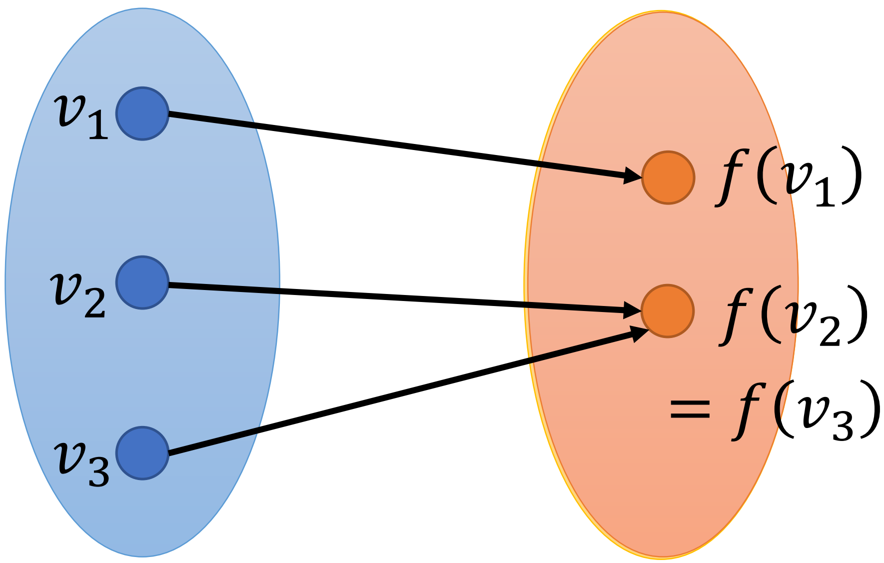
If a matrix A is 矮胖, it cannot be one-to-one.

The reverse is not true.

If a matrix A is one-to-one, its columns are independent.

Onto

- A function f is onto



Co-domain = range

$f(x) = b$ always have solution

If co-domain is “larger” than the domain, f cannot be onto.

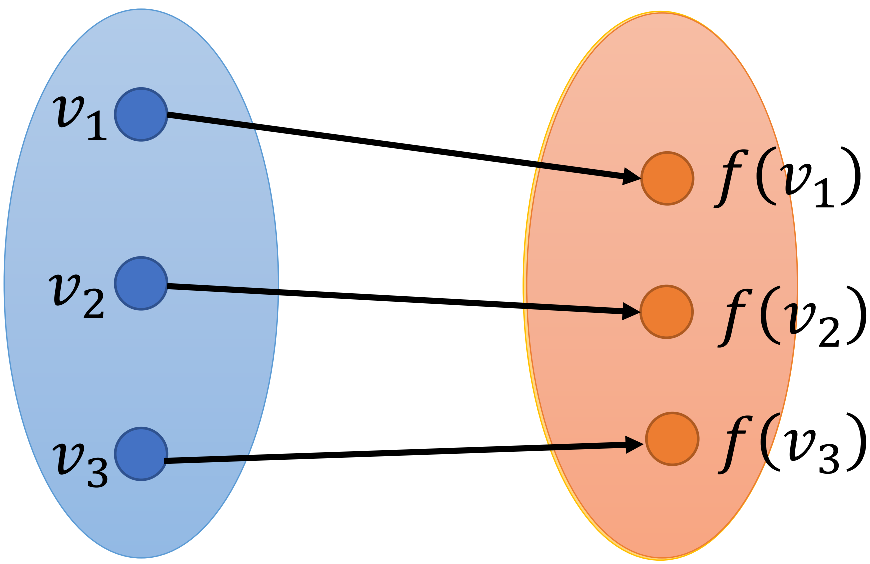
If a matrix A is 高瘦, it cannot be onto.

The reverse is not true.

If a matrix A is onto, rank A = no. of rows.

One-to-one and onto

- A function f is one-to-one and onto



The domain and co-domain must have "the same size".

The corresponding matrix A is square.

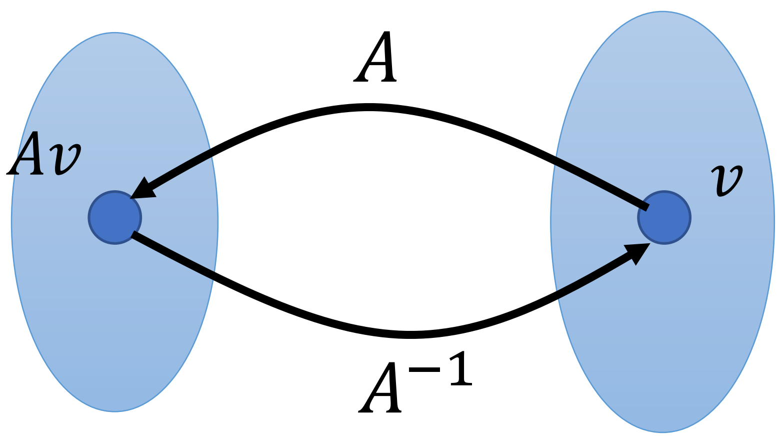
One-to-one ⟷ Onto

在滿足 Square 的前提下，要就都成立，要就都不成立

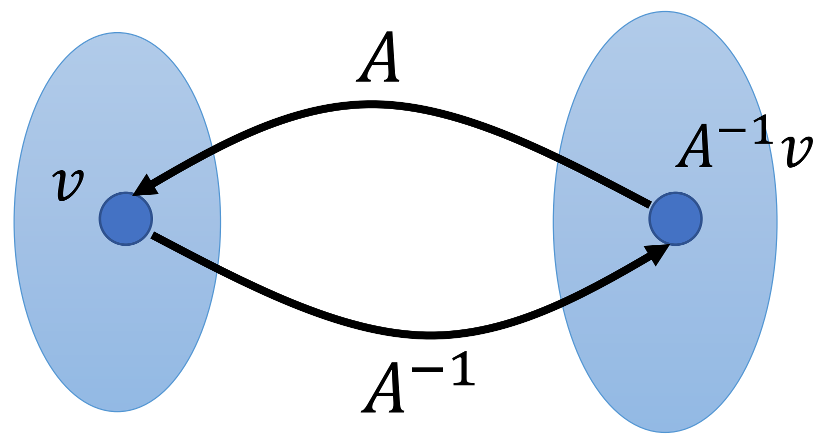
Invertible

An invertible matrix A is always square.

- A is called invertible if there is a matrix B such that $AB = I$ and $BA = I$ ($B = A^{-1}$)



A must be one-to-one



A must be onto

(不然 A^{-1} 的 input 就會有限制)

Invertible

- Let A be an n x n matrix.
  - Onto → One-to-one → invertible
    - The columns of A span R^n
    - For every b in R^n, the system Ax=b is consistent
    - The rank of A is the number of rows
  - One-to-one → Onto → invertible
    - The columns of A are linear independent
    - The rank of A is the number of columns
    - The nullity of A is zero
    - The only solution to Ax=0 is the zero vector
    - The reduced row echelon form of A is I_n

Rank A = n

Invertible

- Let A be an n x n matrix. A is invertible if and only if
  - The reduced row echelon form of A is I_n

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 4 & 8 \end{bmatrix} \xrightarrow[\text{RREF}]{} I_n$$

Invertible

$$B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \xrightarrow[\text{RREF}]{} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Not Invertible

Summary

- Let A be an n x n matrix. A is invertible if and only if
  - The columns of A span R^n
  - For every b in R^n, the system Ax=b is consistent
  - The rank of A is n
  - The columns of A are linear independent
  - The only solution to Ax=0 is the zero vector
  - The nullity of A is zero
  - The reduced row echelon form of A is I_n
  - A is a product of elementary matrices
  - There exists an n x n matrix B such that BA = I_n
  - There exists an n x n matrix C such that AC = I_n

onto

One-to-one

=

square matrix

Invertible

An n x n matrix A is invertible.

⇕

The reduced row echelon form of A is I_n

⇕

A is a product of elementary matrices

R=RREF(A)=I_n

$$E_k \cdots E_2 E_1 A = I_n$$

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} I_n$$

$$= E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$

Invertible

An n x n matrix A is invertible.

There exists an n x n matrix B such that BA = I_n

?

The only solution to Ax=0 is the zero vector

If $Av = 0$, then ....

$$BA = I_n \Rightarrow v = 0$$

$$BAv = 0 \qquad I_n v = v$$

Invertible

An n x n matrix A is invertible.

There exists an n x n matrix C such that AC = I_n

?

For every b in R^n, Ax=b is consistent

For any vector b,

$$AC = I_n$$

ACb $\quad\quad$ $I_n b = \mathrm{b}$

Cb is always a solution for b

Summary

- Let A be an n x n matrix. A is invertible if and only if

onto

- The columns of A span R^n
- For every b in R^n, the system Ax=b is consistent
- The rank of A is n

One-to-one

- The columns of A are linear independent
- The only solution to Ax=0 is the zero vector
- The nullity of A is zero
- The reduced row echelon form of A is I_n

- A is a product of elementary matrices
- There exists an n x n matrix B such that BA = I_n
- There exists an n x n matrix C such that AC = I_n

= square matrix

Questions

- If A and B are matrices such that $AB=I_n$ for some n, then both A and B are invertible.
- For any two n by n matrices A and B, if $AB=I_n$, then both A and B are invertible.

Inverse of a Matrix

Inverse of
General invertible matrices

2 X 2 Matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

Find e, f, g, h

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, A is not invertible.

Algorithm for Matrix Inversion

- Let A be an n x n matrix. A is invertible if and only if
  - The reduced row echelon form of A is I_n

$$\underbrace{E_k \cdots E_2 E_1}_{A^{-1}} A = R = I_n$$

$$A^{-1} = E_k \cdots E_2 E_1$$

Algorithm for Matrix Inversion

- Let A be an n x n matrix. Transform [A I_n] into its RREF [R B]
  - R is the RREF of A
  - B is an nxn matrix (not RREF)
- If R = I_n, then A is invertible
  - B = A^{-1}

$$E_k \cdots E_2 E_1 [A \quad I_n]$$

$$= [\underbrace{R}_{I_n} \quad \underbrace{E_k \cdots E_2 E_1}_{A^{-1}}]$$

Algorithm for Matrix Inversion

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 4 & 8 \end{bmatrix} \xrightarrow[\text{RREF}]{} I_n$$

Invertible

$$\begin{bmatrix} A & I_3 \end{bmatrix} = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 6 & 0 & 1 & 0 \\ 3 & 4 & 8 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & -2 & -1 & -3 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & -7 & 2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7 & -2 & -1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -20 & 6 & 3 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7 & -2 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -16 & 4 & 3 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7 & -2 & -1 \end{array}\right]$$

$$A^{-1}$$

Algorithm for Matrix Inversion

- Let A be an n x n matrix. Transform [A I_n] into its RREF [R B]
  - R is the RREF of A
  - B is a nxn matrix (not RREF)
- If R = I_n, then A is invertible
  - B = A^{-1}
- To find A^{-1}C, transform [A C] into its RREF [R C']
  - C' = A^{-1}C

$$E_k \cdots E_2 E_1 [A \quad C] = [\underbrace{R}_{I_n} \quad \overbrace{\underbrace{E_k \cdots E_2 E_1}_{A^{-1}} C}^{A^{-1}C}]$$

Acknowledgement

- 感謝 周昀 同學發現投影片 上的錯誤

Appendix

2 X 2 Matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{bmatrix} a \\ c \end{bmatrix} \neq k \begin{bmatrix} b \\ d \end{bmatrix} \qquad \frac{a}{b} \neq \frac{c}{d}$$

$$ad \neq bc \qquad ad - bc \neq 0$$

$$A^{-1} = \begin{bmatrix} e & f \\ g & f \end{bmatrix}$$

Find e, f, g, h

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$